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A Generalization of a Theorem of Pringsheim

HANS DENK

Reschweg, D-7900 Ulm-Lehr, West Germany

AND

MAX RIEDERLE

Eberhardtstr. 14/3, D-7900Ulm, West Germany

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We consider noncommutative continued fractions of the form

$$b_0 + a_1(b_1 + a_2(b_2 + a_3(...)^{-1} c_3)^{-1} c_2)^{-1} c_1, \tag{1}$$

where a_n, b_n and c_n are elements of some Banach algebra B and b_n^{-1} exists. Such expressions play an important role in the numerical investigation of various problems in theoretical physics and in applied mathematics, but up to now their convergence was not studied in the general case. In this paper we prove a theorem which is an extension of a wellknown theorem of Pringsheim and, in particular, guarantees the convergence of (1) under the following hypotheses:

$$\|a_{n+1} b_{n+1}^{-1}\| \cdot \|c_{n+1} b_n^{-1}\| \leqslant \tfrac{1}{4} \text{ for all } n \in \mathbb{N}_0,$$

$$4^n \prod_{\nu=0}^{n} \|a_{\nu+1} b_{\nu+1}^{-1}\| \cdot \|c_{\nu+1} b_\nu^{-1}\| \xrightarrow[n \to \infty]{} 0.$$

As an application, we give a generalization of a theorem of van Vleck. The paper closes with an extensive bibliography.

1. INTRODUCTION

In this paper, we investigate expressions of the form

$$b_0 + a_1(b_1 + a_2(b_2 + a_3(...)^{-1} c_3)^{-1} c_2)^{-1} c_1, \tag{1}$$

where (a_n), (b_n) and (c_n) are sequences of elements of some Banach algebra B with identity e. In analogy to the case $B = \mathbb{C}$, these expressions are called continued fractions. They are, in general, noncommutative.

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Noncommutative continued fractions of the more specialized forms

$$b_0 + a_1(b_1 + a_2(b_2 + a_3(...)^{-1})^{-1})^{-1}$$

or (2)

$$b_0 + (b_1 + (b_2 + (...)^{-1} c_3)^{-1} c_2)^{-1} c_1$$

are quite important in applied mathematics. For example, they occur as solutions of the equation $x^2 - xc - d = 0$ in Banach algebras [35, 37], which is of particular interest in the stability theory of differential equations. Moreover, they are used in the interpolation theory [44], for the calculation of square roots of matrices [26] and for other problems in numerical analysis. Furthermore, they play an important role in control theory. In fact, the transfer function matrix which describes the inputs and the outputs of a system may be expanded into a noncommutative continued fraction [25, 26]. A collection of other interesting applications may be found in [44, 45]; for continued fractions of a related type compare also McFarland [51].

The large spectrum of applications suggests the investigation of the convergence properties of expressions of type (2). The first results in this direction are due to Fair [36–38], Hayden [40], Negoescu [41–43] and Wynn [46]. These authors chiefly considered the case $b_n = e$ and stated some convergence theorems closely related to Worpitzky's theorem [8], which guarantees the convergence of complex continued fractions under the hypotheses $|a_n| \leqslant \frac{1}{4}$ and $b_{n-1} = c_n = e$ for all $n \in \mathbb{N}$. The best result is due to Negoescu [41], who proved the complete analogon. Unfortunately, expressions of type (2) cannot be treated by these theorems since each equivalence transformation of them to the case $b_n = e$ seems to produce an expression of type (1), in general. Therefore, it is natural to study these more general expressions.

The investigation of continued fractions of type (1) is natural from another point of view, too. Indeed, they become of increasing interest in physics. Here, in general, B denotes the set of all complex square matrices having a fixed type, and the occurring expressions of type (1) are therefore called matrix continued fractions. For example, they are used for the study of the Brownian motion and the interaction of an atomic system with a laser, for the calculation of eigenvalues and in perturbation theory [9–20]. All these papers use noncommutative continued fractions for numerical purposes without investigating their convergence.

In this paper, we prove a convergence theorem for continued fractions of type (1) which is closely related to a theorem of Pringsheim (cf. Theorem 1). Our proof is based on the study of operators H_n having the form $H_n(x) = b_n + a_{n+1} x^{-1} c_{n+1}$ which allow a direct approach to the convergents (cf. Definition 1). In particular, we do not investigate the numerators and denominators of the convergents because, in general, there is no analogon to the classical recurrence relations. It turns out that the convergence of most of the matrix continued fractions occurring in the above-mentioned problems in physics may be decided by our theorem. This will be shown in a separate paper. Further, we study continued fractions whose entries are functions with range in B and prove a theorem concerning their uniform convergence and their continuity. This theorem generalizes a result of van Vleck [6]. Finally, we tried to collect in the bibliography the papers concerning continued fractions in Banach algebras.

2. A Convergence Theorem

Throughout this paper, K denotes a field with a valuation ψ and B a Banach algebra over K with norm $\|\cdot\|$, identity e and $\|e\| = 1$. Moreover, we write B^* for the set of all the invertible elements of B.

Definition 1. Let (a_n), (b_n) and (c_n) be sequences with $a_n, c_n \in B$ for all $n \in \mathbb{N}$ and $b_n \in B^*$ for all $n \in \mathbb{N}_0$. We put $H_\nu: B^* \to B$, $H_\nu(x) = b_\nu + a_{\nu+1} x^{-1} c_{\nu+1}$. Then the expression (1) has two meanings: first, it denotes the sequence (r_n) defined by $r_0 := b_0$, $r_n := H_0 \circ \cdots \circ H_{n-1}(b_n)$ for $n \in \mathbb{N}$; second, if the sequence (r_n) is well-defined for all n greater than some number n_0 and tends to some limit x_0, we write

$$x_0 = b_0 + a_1(b_1 + a_2(b_2 + a_3(...)^{-1} c_3)^{-1} c_2)^{-1} c_1.$$

Obviously, we have

$$r_n = b_0 + a_1(b_1 + a_2(b_2 + \cdots (b_{n-1} + a_n b_n^{-1} c_n)^{-1} \cdots)^{-1} c_2)^{-1} c_1;$$

therefore, it is natural to call r_n the nth convergent of (1).

We state some special cases of Definition 1:

1. If K is complete with respect to ψ, we may put $B = K$, $\|\cdot\| = \psi$ and $c_n = e$. This leads for nonarchimedean valuations to the continued fractions considered by Butuc [30]. Moreover, if $K = \mathbb{C}$ or $K = \mathbb{R}$ and ψ is the absolute value, we obtain the ordinary continued fractions.
2. If B is an arbitrary complex Banach algebra and $c_n = b_n = e$, we get the continued fractions studied by Fair [36–38], Hayden [40] and Negoescu [41–43].
3. If B is the set of all the matrices of type (q, q) with complex elements, we obtain matrix continued fractions. They were studied by Pfluger [24]; moreover, as indicated in the introduction, they occur in control theory and in physics.

4. If Q is the quaternion algebra, we may treat continued fractions in Q if we put $B = Q$, $K = \mathbb{R}$, $\psi = |\cdot|$ and use the usual norm of Q. In this case, the norm even satisfies $\|a \cdot b\| = \|a\| \cdot \|b\|$ for all $a, b \in Q$. For another Banach algebra with this property, consider l_1 (cf. [2, p. 68]).

THEOREM 1. *Let (p_n) be a sequence of real numbers with $p_n > 1$ for all $n \in \mathbb{N}_0$. Suppose that*

$$\|a_{n+1} b_{n+1}^{-1}\| \cdot \|c_{n+1} b_n^{-1}\| \leqslant (p_n - 1)/(p_n p_{n+1})$$

for all $n \in \mathbb{N}_0$. Furthermore, let

$$(1/p_{n+1}) \prod_{\nu=0}^{n} \|a_{\nu+1} b_{\nu+1}^{-1}\| \cdot \|c_{\nu+1} b_\nu^{-1}\| \cdot p_{\nu+1}^2 \xrightarrow[n \to \infty]{} 0.$$

(a) *Then the continued fraction* (1) *is convergent to some limit $x_0 \in \{x: \|x b_0^{-1} - e\| \leqslant (p_0 - 1)/p_0\}$;*

(b) $\|r_n - x_0\| \leqslant (\|b_0\|/p_{n+1}) \prod_{\nu=0}^{n} \|a_{\nu+1} b_{\nu+1}^{-1}\| \cdot \|c_{\nu+1} b_\nu^{-1}\| p_{\nu+1}^2$.

Remarks. 1. Part (a) of Theorem 1 is quite similar to a theorem of Pringsheim [4]. In fact, if the norm of B has the additional property $\|a \cdot b\| = \|a\| \cdot \|b\|$ for all $a, b \in B$, and if we put $c_n = e$, our hypotheses read

$$\|a_{\nu+1}\|/\|b_\nu b_{\nu+1}\| \leqslant (p_\nu - 1)/(p_\nu p_{\nu+1}) \qquad \text{for all} \quad \nu \in \mathbb{N}_0, \tag{3}$$

$$\prod_{\nu=0}^{n} p_\nu p_{\nu+1} \|a_{\nu+1}\|/\|b_\nu b_{\nu+1}\| \xrightarrow[n \to \infty]{} 0. \tag{4}$$

Obviously, (3) is just Pringsheim's condition, and (4) is an additional hypothesis. It may be interpreted as follows: if $p_n = 2$ for all $n \in \mathbb{N}_0$, the right side of (3) is $\frac{1}{4}$, and condition (4) means that equality cannot always hold in (3). More precisely, (4) is a measure for the deviation of $\frac{1}{4}$ and $\|a_{\nu+1}\|/\|b_\nu b_{\nu+1}\|$ guaranteeing the convergence of (1).

2. For $b_n = c_n = e$, Theorem 1 is due to Hayden [42]. For weaker results compare also Fair [36–38]. Negoescu even proved in [41] that (4) may be omitted under these special assumptions. He also pointed out that the upper bound $\frac{1}{4}$, which occurs in (3) in the case $p_n = 2$, is best possible in some sense.

3. By specifying the sequence (p_n) one may prove analoga to some results given in [3, p. 62]. We omit the details.

Proof of Theorem 1. We may restrict ourselves to the case $b_n = e$ because of the following equivalence transformation: Let $\tilde{r}_n$ denote the nth convergent of the continued fraction $e + \tilde{a}_1(e + \tilde{a}_2(\ldots)^{-1} \tilde{c}_2)^{-1} \tilde{c}_1$, where

$\tilde{a}_{n+1} = a_{n+1} b_{n+1}^{-1}$ and $\tilde{c}_{n+1} = c_{n+1} b_n^{-1}$. Then, with the notation $\tilde{H}_\nu(x) = e + \tilde{a}_{\nu+1} x^{-1} \tilde{c}_{\nu+1}$, a simple induction-type proof shows that

$$H_\nu \circ \cdots \circ H_{n-1}(b_n) = \tilde{H}_\nu \circ \cdots \circ \tilde{H}_{n-1}(e)\, b_\nu$$

for all $n \in \mathbb{N}$ and all $\nu \in \{0, 1, \ldots, n-1\}$, thus

$$r_n = \tilde{r}_n b_0. \tag{5}$$

In order to treat the case $b_n = e$, put $D_n := \{x : \|x - e\| \leqslant (p_n - 1)/p_n\}$ for all $n \in \mathbb{N}_0$. Then $D_n \subset B^*$ and $\|x^{-1}\| \leqslant p_n$ for all $x \in D_n$. Therefore, we have

$$\|H_n(x) - e\| = \|a_{n+1} x^{-1} c_{n+1}\| \leqslant (p_n - 1)/p_n$$

for all $x \in D_{n+1}$. Hence H_n maps D_{n+1} into D_n. This implies that $r_n = H_0 \circ \cdots \circ H_{n-1}(e)$ is well defined and lies in D_0 for all $n \in \mathbb{N}_0$.

Moreover, H_n satisfies a Lipschitz condition on D_{n+1} because we have, for all $x, y \in D_{n+1}$,

$$\begin{aligned}\|H_n(x) - H_n(y)\| &= \|a_{n+1} x^{-1}(y - x) y^{-1} c_{n+1}\| \\ &\leqslant \|a_{n+1}\| \cdot \|c_{n+1}\| p_{n+1}^2 \|x - y\|.\end{aligned}$$

Using this, we obtain, after a short calculation, that

$$\begin{aligned}\|r_{n+m} - r_n\| &= \|H_0 \circ \cdots \circ H_{n-1}(H_n \circ \cdots \circ H_{n+m-1}(e)) - H_0 \circ \cdots \circ H_{n-1}(e)\| \\ &\leqslant (1/p_{n+1}) \prod_{\nu=0}^{n} \|a_{\nu+1}\| \cdot \|c_{\nu+1}\| \cdot p_{\nu+1}^2.\end{aligned} \tag{6}$$

By hypothesis, this sequence tends to zero. Therefore, and since D_0 is closed, (r_n) converges to some limit $x_0 \in D_0$. The last part of the assertion finally follows from (5) and (6), if we let m tend to infinity.

3. Continuity

In this section we want to apply Theorem 1 to expressions of the form

$$b_0(z) + a_1(z)(b_1(z) + a_2(z)(\ldots)^{-1} c_2(z))^{-1} c_1(z), \tag{7}$$

where $a_n(z)$, $b_n(z)$ and $c_n(z)$ are functions from some metric space M into B. We write $r_n(z)$ for the nth convergent of (7), and if the sequence $(r_n(z))$ is convergent for some $z \in M$, we denote its limit by $f(z)$. Such expressions occur, for example, as solutions of the Riccati differential equation in Banach algebras (cf. [39]), in the study of the operator-valued Padé-tables [44] and in the investigation of the ε-algorithm which is used for the acceleration of the convergence of matrix sequences. Moreover, a noncommutative analogon of the quotient–difference algorithm may be obtained [5, 44].

Here, we want to prove a generalization of a theorem of van Vleck [6]:

THEOREM 2. *Let D be an open subset of M such that $a_n(z)$, $b_n(z)$ and $c_n(z)$ fulfill the conditions of Theorem 1 for all $z \in D$. Moreover, suppose that*

$$(\|b_0(z)\|/p_{n+1}) \prod_{\nu=0}^{n} \|a_{\nu+1}(z)\, b_{\nu+1}^{-1}(z)\| \cdot \|c_{\nu+1}(z) b_\nu^{-1}(z)\| \cdot p_{\nu+1}^2$$

tends on D uniformly to zero. Then $(r_n(z))$ is uniformly convergent on D to some function $f(z)$, and $f(z)$ is continuous on D.

Proof. Obviously, it suffices to show that $(r_n(z))$ is uniformly convergent on D and that $r_n(z)$ is continuous on D for all $n \in \mathbb{N}_0$.

In order to prove the first, we note that

$$\begin{aligned}\|r_n(z) - f(z)\| \leqslant (\|b_0(z)\|/p_{n+1}) \prod_{\nu=0}^{n} \|a_{\nu+1}(z)\, b_{\nu+1}^{-1}(z)\| \\ \cdot \|c_{\nu+1}(z)\, b_\nu^{-1}(z)\| \cdot p_{\nu+1}^2\end{aligned}$$

for all $z \in D$ according to part (b) of Theorem 1, and from this we may conclude the uniform convergence of $(r_n(z))$ on D.

In order to prove the continuity of $r_n(z)$, we first consider the case $b_n(z) \equiv e$. Let $D_n := \{x\colon \|x - e\| \leqslant (p_n - 1)/p_n\}$ for all $n \in \mathbb{N}_0$ and $H_{n,z}\colon B^* \to B$, $H_{n,z}(x) = e + a_{n+1}(z)\, x^{-1}\, c_{n+1}(z)$ for all $z \in D$ and all $n \in \mathbb{N}_0$. Then, for $x, y \in D_{n+1}$ and $z \in D$, we obtain

$$\begin{aligned}H_{n,z}(x) - H_{n,z_0}(y) = {} & (a_{n+1}(z) - a_{n+1}(z_0))\, x^{-1} c_{n+1}(z) \\ & + a_{n+1}(z_0)\, x^{-1}(c_{n+1}(z) - c_{n+1}(z_0)) \\ & + a_{n+1}(z_0)\, x^{-1}(y - x)\, y^{-1} c_{n+1}(z_0).\end{aligned}$$

The continuity of $a_{n+1}(z)$ and $c_{n+1}(z)$ in z_0 implies that $a_{n+1}(z)$ and $c_{n+1}(z)$ are bounded in suitably chosen neighbourhoods of z_0. Furthermore, $\|x^{-1}\| \leqslant p_{n+1}$ and $\|y^{-1}\| \leqslant p_{n+1}$. Thus, we may write

$$\|H_{n,z}(x) - H_{n,z_0}(y)\| \leqslant \alpha_n(z) + \beta_n(z_0) \|x - y\|,$$

where $\alpha_n(z) = o(1)$ as z tends to z_0. This implies that

$$\begin{aligned}\|r_n(z) - r_n(z_0)\| &= \|H_{0,z} \circ \cdots \circ H_{n-1,z}(e) - H_{0,z_0} \circ \cdots \circ H_{n-1,z_0}(e)\| \\ &\leqslant \sum_{\nu=0}^{n-1} \alpha_\nu(z) \prod_{\tau=0}^{\nu-1} \beta_\tau(z_0) = o(1)\end{aligned}$$

as z tends to z_0. Therefore, $r_n(z)$ is continuous at z_0. The general case now easily follows from this and (5).

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